LESSON-12

Dimensional Analysis for Free and Forced Convection:

In a number of engineering applications involving flow of fluids over a flat plate, inside and outside of cylinders, heat is exchanged between fluids and solid surfaces. In order to determine heat transfer rate, value of convective heat transfer coefficient must be determined. The following methods are generally used to determine the value of convective heat transfer coefficient.

i) Dimensional Analysis
ii) Solution of Boundary Layer Equations
iii) Analogy between Heat and Momentum Transport

Dimensional analysis:

The method of dimensional analysis was first used by Nusselt to derive mathematical equations for convective heat transfer coefficients for free and forced convection. Dimensional analysis is a mathematical technique which is used to obtain equations governing an unknown physical phenomenon in terms of important parameters influencing that phenomenon. The influencing parameters are organized into dimensionless groups, thereby, reducing the number of influencing parameters. Dimensional analysis for free and forced convection involves following steps

i) Determination of all parameters/variables affecting convective heat transfer coefficient.
ii) Writing influencing parameters in terms of fundamental units of mass, length, time and temperature.
iii) Developing mathematical expressions for convective heat transfer coefficient in terms of fundamental units by using principle of dimensional homogeneity.
iv) Grouping of all influencing parameters into non-dimensional numbers.

The dimensions of various parameters which are important from heat transfer point of view are given in terms of fundamental units of mass, length, time and temperature in Table 1.

Parameter	Symbol	SI Units	Dimensions
Mass	M	Kg	M
Length	L	m	L
Time	T	Second	T
Temperature	θ	K	θ
Heat	Q	Joule	ML^2T^{-2}
Area	A	m^2	L^2
Volume	V	m^3	L^3
Velocity	U, V	m/sec	LT^{-1}
Acceleration	A	m/sec^2	LT^{-2}
Acceleration due to Gravity	G	m/sec^2	LT^{-2}
Force or Resistance	F, R	N	MLT^{-2}
Density	ρ	Kg/m^3	ML^{-3}
Dynamic viscosity	μ	Kg/(m-sec)	$ML^{-1}T^{-1}$
Kinematic viscosity	υ	m^2/sec	L^2T^{-1}
Energy, Work, Heat	E, W	m N	ML^2T^{-2}
Convective heat transfer coefficient	h	$W/(m^2\text{-deg})$	$MT^{-3}\ \theta^{-1}$
Coefficient of volumetric expansion	β	Per deg	θ^{-1}
Specific Heat	C_p, C_v	kJ/(kg-deg)	$L^2T^2\ \theta^{-1}$
Thermal conductivity	K	W/(m-deg)	$MLT^{-3}\ \theta^{-1}$
Thermal resistance per unit area	R_t	m^2-hr- ^oC/kJ	$M^{-1}T^{-3}\ \theta^{-1}$
Thermal Diffusivity	α	m^2/sec	L^2T^{-1}

Methods of Dimensional Analysis:

If number of variables influencing convective heat transfer coefficient are known, then the following two methods can be used to develop a mathematical expression relating the variables with the convective heat transfer coefficient.

i) Rayleigh's Method

ii) Buckingham's π-theorem

However, in application of dimensional analysis for determining convective heat transfer coefficient for free and forced convection, Rayleigh's method will not be used as it has certain limitations that can be overcome by using Buckingham's π-theorem method.

Buckingham's π-Theorem Method

In the Rayleigh's method of dimensional analysis, solution becomes more and more cumbersome and laborious if number of influencing variables become more than the fundamental units (M, L, T and θ) involved in the physical phenomenon.. The use of Buckingham's π-theorem method enables to overcome this limitation and states that if there are 'n' variables (independent and dependent) in a physical phenomenon and if these variables contain 'm' number of fundamental dimensions (M, L, T and θ), then the variables are arranged in to (n-m) dimensionless terms called π-terms.

Buckingham's π-Theorem Method can be applied for forced and free convection processes to determine the heat transfer coefficient.

Dimensional Analysis for Forced Convection

On the basis of experience, it is concluded that forced convection heat transfer coefficient is a function of variables given below in Table -2

S. No.	Variable / Parameter	Symbol	Dimensions
1	Fluid density	ρ	ML^{-3}
2	Dynamic viscosity of fluid	μ	$ML^{-1}T^{-1}$
3	Fluid Velocity	V	LT^{-1}
4	Thermal conductivity of fluid	k	$MLT^{-3}\,\theta^{-1}$
5	Specific heat of fluid	C_p	$L^2T^{-2}\,\theta^{-1}$
6	Characteristic length of heat transfer area	D	L

Therefore, convective heat transfer coefficient is expressed as

$h = f(\rho, \mu, V, k, C_p, D)$ (1)

$f(h, \rho, \mu, V, k, C_p, D) = 0$ (2)

Convective heat transfer coefficient, h is dependent variable and remaining are independent variables.

Total number of variables, n = 7

Number of fundamental units, m = 4

According to Buckingham's π-theorem, number of π-terms is given by the difference of total number of variables and number of fundamental units.

Number of π-terms = (n-m) = 7-4 = 3

These non-dimensional π-terms control the forced convection phenomenon and are expressed as

$f(\pi_1, \pi_2, \pi_3) = 0$ (3)

Each π-term is written in terms of repeating variables and one other variable. In order to select repeating variables, following method should be followed.

- Number of repeating variables should be equal to number of fundamental units involved in the physical phenomenon.
- Dependent variable should not be selected as repeating variable.
- The repeating variables should be selected in such a way that one of the variables should contain a geometric property such as length, diameter or height. Other repeating variable should contain a flow property such as velocity or acceleration and the third one should contain a fluid property such as viscosity, density, specific heat or specific weight.
- The selected repeating variables should not form a dimensionless group.
- The selected repeating variables together must have same number of fundamental dimensions.
- No two selected repeating variables should have same dimensions.

The following repeating variables are selected

i) **Dynamic viscosity, μ having fundamental dimensions $ML^{-1}T^{-1}$**

ii) **Thermal conductivity, k having fundamental dimensions $MLT^{-3}\,\theta^{-1}$**

iii) **Fluid velocity, V having fundamental dimensions LT^{-1}**

iv) **Characteristic length, D having fundamental dimensions L**

Each π-term is expressed as:

$\pi_1 = \mu^a\, k^b, V^c, D^d, h$ (4)

Writing down each term in above equation in terms of fundamental dimensions

$M^0L^0T^0\,\theta^0 = (ML^{-1}T^{-1})^a\,(MLT^{-3}\,\theta^{-1})^b\,(LT^{-1})^c\,(L)^d\,MT^{-3}\,\theta^{-1}$

Comparing the powers of M, we get

$0 = a+b+1$, $a+b= -1$ (5)

Comparing powers of L, we get

$0 = -a+b+c +d$ (6)

Comparing powers of T, we get

$0 = -a- 3b-c -3$ (7)

Comparing powers of θ, we get

$0 = -b -1$,

$b=-1$ (8)

Substituting value of 'b' from equation (8) in equation (5), we get

$a = 0$ (9)

Substituting values of 'a' and 'b' in equation (7), we get

$c = 0$ (10)

Substituting the values of 'a', 'b' and 'c' in equation (6), we get

$d = 1$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (4), we get

$\pi_1 = \mu^{-0}\, k^{-1}, V^0, D^1, h$

$\pi_1 = h\, D / k$ (11)

The second π –term is expressed as

$\pi_2 = \mu^a\, k^b, V^c, D^d, \rho$ (12)

$M^0L^0T^0\,\theta^0 = (ML^{-1}T^{-1})^a\,(MLT^{-3}\,\theta^{-1})^b\,(LT^{-1})^c\,(L)^d\,ML^{-3}$

Comparing the powers of M, we get

$0 = a+b+1,\ a+b = -1$ (13)

Comparing powers of L, we get

$0 = -a + b + c + d - 3$ (14)

Comparing powers of T, we get

$0 = -a - 3b - c$ (15)

Comparing powers of θ, we get

$0 = -b$,

$b=0$ (16)

Substituting value of 'b' from equation (16) in equation (13), we get

$a = -1$ (17)

Substituting values of 'a' and 'b' in equation (15), we get

$c = 1$ (18)

Substituting the values of 'a', 'b' and 'c' in equation (14), we get

$d = 1$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (12), we get

$\pi_2 = \mu^{-1} k^0, V^1, D^1, \rho$

$\pi_2 = \rho\ VD / \mu$ (19)

The third π –term is expressed as

$\pi_3 = \mu^a k^b, V^c, D^d, C_p$ (20)

$M^0L^0T^0\ \theta^0 = (ML^{-1}T^{-1})^a\ (MLT^{-3}\ \theta^{-1})^b\ (LT^{-1})^c\ (L)^d\ L^2T^{-2}\ \theta^{-1}$

Comparing the powers of M, we get

$0 = a+b,\ a+b = 0$ (21)

Comparing powers of L, we get

$0 = -a + b + c + d + 2$ (22)

Comparing powers of T, we get

$0 = -a - 3b - c - 2$ (23)

Comparing powers of θ, we get

$0 = -b - 1$,

$b=-1$ (24)

Substituting value of 'b' from equation (24) in equation (21), we get

$a = 1$ (25)

Substituting values of ‘a’ and ‘b’ in equation (23), we get

$c = 0$ (26)

Substituting the values of ‘a’, ‘b’ and ‘c’ in equation (22), we get

$d = 0$

Substituting the values of ‘a’, ‘b’, ‘c’ and ‘d’ in equation (12), we get

$\pi_3 = \mu^1\, k^{-1}, V^0, D^0, C_p$

$\pi_3 = \mu\, C_p/k$ (27)

Substituting the values of π_1, π_2, π_3 in equation (3), we get

$f(h\, D / k, \rho\, VD / \mu, \mu\, C_p/k) = 0$

$h\, D / k = \varphi(\rho\, VD / \mu, \mu\, C_p/k)$

$Nu = \varphi(Re, Pr)$

The above correlation is generally expressed as

$$Nu = C\, (Re)^a\, (Pr)^b$$

The constant C and exponents ‘a’ and ‘b’ are determined through experiments.

Dimensional Analysis for Free Convection:

In free convection heat transfer process, convective heat transfer coefficient depends upon the same parameters/variable as in case of forced convection except velocity of fluid. It is on account of the fact that in free convection motion of fluid occurs due to difference in density of various layers of fluid caused by temperature difference whereas in case of forced convection motion of fluid is caused by an external source. The fluid velocity in case of free convection depends upon the following parameters;

i) **Temperature difference between solid surface and bulk fluid, ΔT**
ii) **Acceleration due to gravity, g**
iii) **Coefficient of volumetric expansion of fluid, β**

The change in the volume when temperature changes can be expressed as

$dV = V_1\, \beta\, (T_2 - T_1)$

where

dV - change in volume (m^3)

$= V_2 - V_1$

β = Coefficient of volumetric expansion of fluid, ($m^3/m^3\, {}^oC$)

T_2 - Final temperature (${}^oC$)

T_1 - Initial temperature (°C)

Therefore, free convection heat transfer coefficient is a function of variables given in Table 3

Table 3

S. No.	Variable	Symbol	Dimensions
1	Fluid density	ρ	ML^{-3}
2	Dynamic viscosity of fluid	μ	$ML^{-1}T^{-1}$
3	Thermal conductivity of fluid	k	$MLT^{-3}\ \theta^{-1}$
4	Specific heat of fluid	C_p	$L^2T^{-2}\ \theta^{-1}$
5	Characteristic length of heat transfer area	D	L
6	Temperature difference between surface and bulk fluid	ΔT	θ
7	Coefficient of volumetric expansion	β	θ^{-1}
8	Acceleration due to gravity	g	LT^{-2}

Therefore, convective heat transfer coefficient is expressed as

$$h = f(\rho, \mu, k, C_p, D, \Delta T, \beta, g) \quad (28)$$

However, in free convection, ($\Delta T\ \beta\ g$) will be treated as single parameter as the velocity of fluid particles is a function of these parameters. Therefore, equation (28) can be expressed as

$$f(h, \rho, \mu, k, C_p, D, (\Delta T\ \beta\ g)) = 0 \quad (29)$$

Convective heat transfer coefficient, h is the dependent variable and remaining are independent variables.

Total number of variables, n = 7

Number of fundamental units, m = 4

According to Buckingham's π-theorem, number of π-terms is given by the difference of total number of variables and number of fundamental units.

Number of π-terms = (n-m) = 7-4 = 3

These non-dimensional π-terms control the forced convection phenomenon and are expressed as

$$f(\pi_1, \pi_2, \pi_3) = 0 \quad (30)$$

Each π-term is written in terms of repeating variables and one other variable and the following repeating variables are selected

i) **Dynamic viscosity, μ having fundamental dimensions $ML^{-1}T^{-1}$**

ii) **Thermal conductivity, k having fundamental dimensions $MLT^{-3}\theta^{-1}$**

iii) **Fluid density, ρ having fundamental dimensions ML^{-3}**

iv) **Characteristic length, D having fundamental dimensions L**

Each π-term is expressed as:

$\pi_1 = \mu^a k^b, \rho^c, D^d, h$ (31)

Writing down each term in above equation in terms of fundamental dimensions

$M^0L^0T^0\theta^0 = (ML^{-1}T^{-1})^a (MLT^{-3}\theta^{-1})^b (ML^{-3})^c (L)^d MT^{-3}\theta^{-1}$

Comparing the powers of M, we get

$0 = a+b+c+1, a+b+c= -1$ (32)

Comparing powers of L, we get

$0 = -a+b+c +d$ (33)

Comparing powers of T, we get

$0 = -a- 3b-c -3$ (34)

Comparing powers of θ, we get

$0 = -b -1$,

$b=-1$ (35)

Substituting value of 'b' from equation (35) in equation (32), we get

$a = 0$ (36)

Substituting values of 'a' and 'b' in equation (34), we get

$c = 0$ (37)

Substituting the values of 'a', 'b' and 'c' in equation (33), we get

$d = 1$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (31), we get

$\pi_1 = \mu^a k^b, \rho^c, D^d, h$

$\pi_1 = h D / k$ (38)

The second π –term is expressed as

$\pi_2 = \mu^a k^b, \rho^c, D^d, C_p$ (39)

$M^0L^0T^0\theta^0 = (ML^{-1}T^{-1})^a (MLT^{-3}\theta^{-1})^b (ML^{-3})^c (L)^d L^2T^{-2}\theta^{-1}$

Comparing the powers of M, we get

$0 = a+b+c$ (40)

Comparing powers of L, we get

$0 = -a+ b-3c +d +2$ (41)

Comparing powers of T, we get

$0 = -a- 3b-2$ (42)

Comparing powers of θ, we get

$0 = -b-1, \quad b=-1$ (43)

Substituting value of 'b' from equation (43) in equation (40), we get

$a = 1$ (44)

Substituting values of 'a' and 'b' in equation (42), we get

$c = 0$ (45)

Substituting the values of 'a', 'b' and 'c' in equation (41), we get

$d = 0$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (39), we get

$\pi_2 = \mu^1\ k^{-1}, \rho^0 , D^0, C_p$

$\pi_2 = \mu\ C_p / k$ = Prandtl Number = Pr (46)

The third π –term is expressed as

$\pi_3 = \mu^a\ k^b, \rho^c, D^d, ((\Delta T\ \beta\ g)$ (47)

$M^0L^0T^0\ \theta^0 = (ML^{-1}T^{-1})^a\ (MLT^{-3}\ \theta^{-1})^b\ (ML^{-3})^c\ (L)^d\ (\theta^{-1}LT^{-2}\ \theta^1)$

$M^0L^0T^0\ \theta^0 = (ML^{-1}T^{-1})^a\ (MLT^{-3}\ \theta^{-1})^b\ (ML^{-3})^c\ (L)^d\ (LT^{-2})$

Comparing the powers of M, we get

$0 = a+b+c, a+b+c= 0$ (48)

Comparing powers of L, we get

$0 = -a+b-3c +d +1$ (49)

Comparing powers of T, we get

$0 = -a- 3b-2$ (50)

Comparing powers of θ, we get

$0 = -b, b=0$ (51)

Substituting value of 'b' from equation (51) in equation (48), we get

$a = -2$ (52)

Substituting values of 'a' and 'b' in equation (50), we get

$c = 2$ (53)

Substituting the values of 'a', 'b' and 'c' in equation (49), we get

$d = 3$

Substituting the values of 'a', 'b', 'c' and'd' in equation (47), we get

$\pi_3 = \mu^{-2}\, k^0, \rho^2, D^3, (\Delta T\ \beta\ g)$

$\pi_3 = \rho^2\, D^3\, (\Delta T\ \beta\ g) / \mu^2$

$= D^3\, (\Delta T\ \beta\ g) / \upsilon^2$ (54)

Substituting the values of π_1, π_2, π_3 in equation (30), we get

$f(h\, D / k,\ \mu\, C_p/k,\ D^3\, (\Delta T\ \beta\ g) / \upsilon^2\,) = 0$

$h\, D / k = \varphi(\mu\, C_p/k,\ D^3\, (\Delta T\ \beta\ g) / \upsilon^2\,)$

$Nu = \varphi(Pr, Gr)$ as $Gr = D^3\, (\Delta T\ \beta\ g) / \upsilon^2$ (55)

The above correlation is generally expressed as

$Nu = C\, (Pr)^a\, (Gr)^b$ (56)

The constant C and exponents a and b are determined through experiments.

REVIEW QUESTIONS:

Q.1 Convective heat transfer coefficient can be determined by using

a) Dimensional analysis method

b) Solution of boundary layer equations

c) Analogy between heat and momentum transport

d) All the above

Q.2 Fundamental dimensions of convective heat transfer coefficient are

a) $MT^{-1}\,\theta^{-1}$

b) $ML^{-3}\,\theta^{-1}$

c) $MT^{-3}\,\theta^{-1}$

d) $M^{-3}\,L^1\,\theta^{-1}$

Q.3 In Buckingham's π-theorem method, number of π-terms is equal to

a) Product of independent and dependent variables

b) Difference of total number of variables and number of fundamental dimensions involved

c) Sum of independent and dependent varialbes

d) Differnce of independent and dependent variables

Q.4 In Buckingham's π-theorem method, number of repeating variables is equal to

a) Differnce of independent and dependent variables

b) Number of independen variables

c) Three

d) number of fundamental dimensions involved

Q.5 Fundamental dimenions of specific heat are

a) $ML^{-2}\,T^{-1}$

b) $MT^{-2}\,\theta^{-1}$

c) $L^2T^2\,\theta^{-1}$

d) $ML^{-2}\,\theta^{-1}$