LESSON-15

Laminar Forced Convection on a Flat Plate:

Consider a fluid flowing over a flat plate with a velocity U and temperature T_f. Let us consider a control volume at a distance 'x' from leading edge of the plate having thickness dx as shown in Figure 1. Following assumptions have been made in order to calculate heat conducted into laminar boundary layer:

i) Thermo-physical properties of the fluid such as thermal conductivity k, specific heat C_p and density ρ remains constant for the range of the temperature

ii) Heating of the plate starts from a distance x_o from leading edge of the plate. Within the initial length x_o, plate temperature is equal to that of the fluid and there is only hydrodynamic boundary layer and no thermal boundary layer exists. Thermal boundary layer starts developing beyond length x_o and keeps growing.

iii) Width of the plate is considered to be unity.

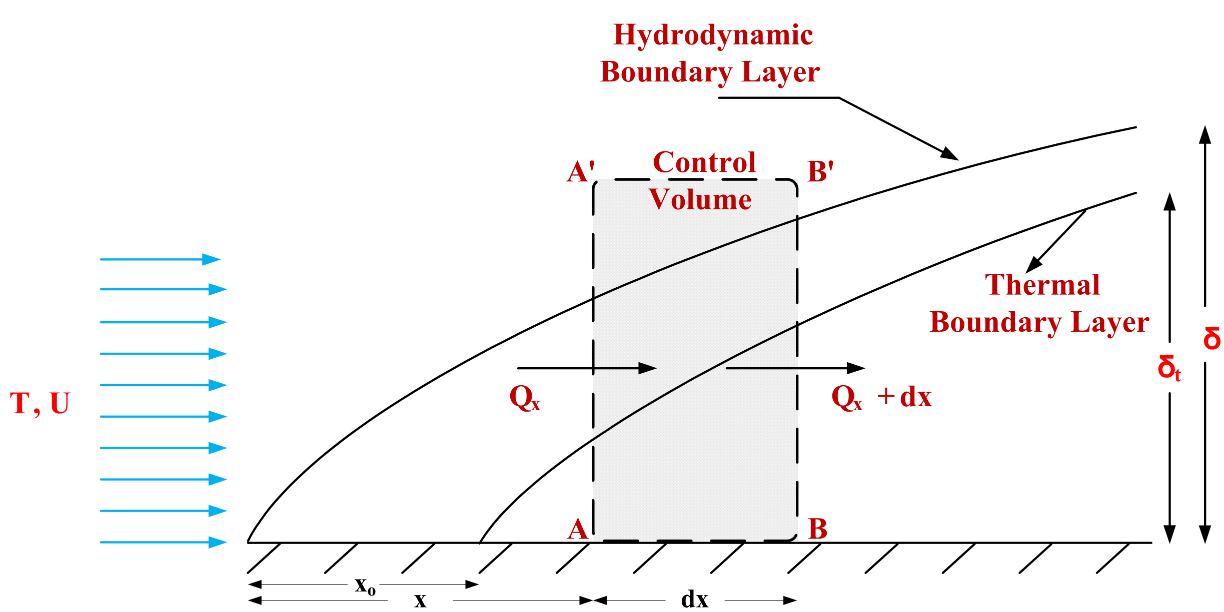

Figure 1

Mass of fluid entering into control volume through left face AA'

$$=\int_0^H \rho u\, dy \tag{1}$$

Mass of fluid leaving control volume through right face BB'

$$=\int_0^H \rho u\, dy + \frac{\partial}{\partial x}\left[\int_0^H \rho u\, dy\right] dx \tag{2}$$

Mass of fluid entering from top face A'B' of control volume

$$=\frac{\partial}{\partial x}\left[\int_0^H \rho u\, dy\right] dx \tag{3}$$

Heat Influx through face AA'

$$Q_x = mass \times specific\, heat \times temperature$$

$$Q_x = \int_0^H \rho u\, dy \times C \times T$$

$$= \rho C \int_0^H u\, T dy \tag{4}$$

Heat efflux through face BB'

$$Q_{x+dx} = \rho C \int_0^H u\, T + \frac{\partial}{\partial x}\left[\rho C \int_0^H u\, T dy\right] dx \tag{5}$$

The upper face A'B' of the control volume is out of the thermal boundary layer and there temperature is constant and is equal to 'T_f'. Therefore, energy influx is

$$Q_h = \frac{\partial}{\partial x}\left[\int_0^H \rho u\, dy\right] dx\, C\, T_f \tag{6}$$

Heat is conducted in to the lower face of the control volume at the rate

$$Q_c = -kA\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$Q_c = -kdx \times 1\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{7}$$

An energy balance of the control volume gives:

$$\rho C\int_0^H uTdy + \frac{\partial}{\partial x}\left[\rho C\int_0^H uT_f\,dy\right]dx - kdx\left(\frac{\partial T}{\partial y}\right)_{y=0} = \rho C\int_0^H uTdy + \frac{\partial}{\partial x}\left[\rho C\int_0^H uT\,dy\right]dx \quad (8)$$

Rearranging equation (8), we get,

$$\frac{d}{dx}\left[\int_0^H u\left(T_f - T\right)dy\right] = \frac{k}{\rho C}\left(\frac{dT}{dy}\right)_{y=0}$$

$$\frac{d}{dx}\left[\int_0^H u\left(T_f - T\right)dy\right] = \alpha\left(\frac{dT}{dy}\right)_{y=0} \quad (9)$$

Where α represents thermal diffusivity

Equation (9) represents the integral equation for the boundary layer for constant properties and constant free stream temperature T_f.

The net viscous work done with in the control volume is given by the equation

$$\mu\int_0^H \frac{\partial^2 u}{\partial y^2}dxdy \quad (10)$$

If the net viscous work done is also considered in the energy balance, then the integral equation would become

$$\frac{d}{dx}\left[\int_0^H u\left(T_f - T\right)dy\right] + \frac{\mu}{\rho C}\int_0^H \frac{\partial^2 u}{\partial y^2}dy = \frac{k}{\rho C}\left(\frac{dT}{dy}\right)_{y=0} \quad (10)$$

The term related to viscous work is generally very small and is usually neglected.

To develop an expression for convective heat transfer coefficient for laminar flow over a plate, cubic velocity and temperature distribution in integral boundary layer equation will be used.

i) The temperature distribution with in the boundary layer is given as

$$\frac{u}{u_f} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \quad (11)$$

ii) Temperature distribution with in the boundary layer satisfies the conditions;

At y = 0, T = T and $\frac{d^2T}{dy^2} = 0$

At y = δ_t, $\frac{dT}{dy} = 0$ and $T = T_f$

These boundary conditions have same form as that of $\frac{u}{u_f}$. Therefore, when these are fitted to a cubic polynomial, we get

$$\frac{\theta}{\theta_f}=a+b\left(\frac{y}{\delta_t}\right)+c\left(\frac{y}{\delta_t}\right)^2+d\left(\frac{y}{\delta_t}\right)^3 \tag{12}$$

Temperature distribution acquires the form

$$\frac{\theta}{\theta_f}=\frac{T-T_s}{T_f-T_s}=\frac{3}{2}\left(\frac{y}{\delta_t}\right)-\frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \tag{13}$$

Multiplying and dividing right hand side of the integral equation (9) by $u_f\left(T_f-T_s\right)$, we can write

$$\alpha\left(\frac{dT}{dy}\right)_{y=0}=u_f\left(T_f-T_s\right)\frac{d}{dx}\left[\int_0^H \frac{u\left(T_f-T\right)}{u_f\left(T_f-T_s\right)}dy\right]$$

$$=u_f\left(T_f-T_s\right)\frac{d}{dx}\left[\int_0^H \frac{u}{u_f}\left(1-\frac{\left(T-T_s\right)}{\left(T_f-T_s\right)}\right)dy\right]$$

Using equations (11) and (13), we can write

$$\alpha\left(\frac{dT}{dy}\right)_{y=0}=u_f\left(T_f-T_s\right)\frac{d}{dx}\left[\int_0^H\left\{\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}\left\{1-\frac{3}{2}\left(\frac{y}{\delta_t}\right)-\frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right\}dy\right] \tag{14}$$

For most of the gases, thermal boundary layer is thinner than the hydrodynamic boundary layer $\delta_t < \delta$. Therefore the upper limit of integration in equation (14) has been changed to δ_t as for $y > \delta_t$, the integral will become zero. Let 'r' represents thickness ratio and it is equal to δ_t/δ.

Upon integrating equation (14) between limits, we get

$$\alpha\left(\frac{dT}{dy}\right)_{y=0}=u_f\left(T_f-T_s\right)\frac{d}{dx}\left[\delta\left(\frac{3}{20}r^2-\frac{3}{280}r^4\right)\right] \tag{15}$$

As $\delta_t < \delta$, $r < 1$, therefore, term involving r^4 may be neglected

$$\alpha\left(\frac{dT}{dy}\right)_{y=0}=\frac{3}{20}u_f\left(T_f-T_s\right)\frac{d}{dx}\left[\delta r^2\right] \tag{16}$$

Using temperature distribution equation (13), we can write

$$\frac{T-T_s}{T_f-T_s}=\frac{3}{2}\left(\frac{y}{\delta_t}\right)-\frac{1}{2}\left(\frac{y}{\delta_t}\right)^3$$

$$T=T_s+\left(T_f-T_s\right)\left[\frac{3}{2}\left(\frac{y}{\delta_t}\right)-\frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]$$

$$\frac{dT}{dy}=\left(T_f-T_s\right)\left[\frac{3}{2\delta_t}-\frac{3}{2\delta_t^3}y^2\right]$$

$$\left(\frac{dT}{dy}\right)_{y=0}=\frac{3\left(T_f-T_s\right)}{2\delta_t}=\frac{3\left(T_f-T_s\right)}{2r\delta} \tag{17}$$

Substituting the value of $\left(\frac{dT}{dy}\right)_{y=0}$ from equation (17) in equation (16), we get

$$\frac{3}{2}\alpha\frac{\left(T_f-T_s\right)}{r\delta}=\frac{3}{20}u_f\left(T_f-T_s\right)\frac{d}{dx}\left[\delta r^2\right]$$

$$\alpha=\frac{u_f}{10}\left(r\delta\right)\frac{d}{dx}\left[\delta r^2\right]$$

$$=\frac{u_f}{10}\left(r\delta\right)\left(2r\delta\frac{dr}{dx}+r^2\frac{d\delta}{dx}\right)$$

$$\alpha=\frac{u_f}{10}\left(2r^2\delta^2\frac{dr}{dx}+\delta r^3\frac{d\delta}{dx}\right) \tag{18}$$

Using the hydrodynamic boundary layer equations

$$\delta\frac{d\delta}{dx}=\frac{140}{13}\frac{v}{u_f}\ and\ \delta^2=\frac{280}{13}\frac{vx}{u_f}$$

Substituting these values in equation (18), we get

$$\alpha=\frac{u_f}{10}\left(2r^2\frac{280\,vx}{13u_f}\frac{dr}{dx}+r^3\frac{140\,v}{13u_f}\right)$$

$$r^3+4r^2x\frac{dr}{dx}=\frac{13}{14}\frac{\alpha}{v} \tag{19}$$

Equation (19) is a linear differential equation of first order in r^3 and general solution for it is given as

$$r^3=C\,x^{-\frac{3}{4}}+\frac{13}{14}\frac{\alpha}{v} \tag{20}$$

The constant C is determined by using the boundary condition

At x=xₒ, $r^3 = \left(\frac{\delta_t}{\delta}\right)^3 = 0$

$$0 = C\,x^{-\frac{3}{4}} + \frac{13}{14}\frac{\alpha}{v}, \; C = \frac{13}{14}\frac{\alpha}{v}x_o^{3/4} \tag{21}$$

Substituting the value of C from equation (21) into equation (20), we get

$$r^3 = -\frac{13}{14}\frac{\alpha}{v}x_o^{3/4}x^{-\frac{3}{4}} + \frac{13}{14}\frac{\alpha}{v}$$

$$r^3 = \frac{13}{14}\frac{\alpha}{v}\left[1-\left(\frac{x_o}{x}\right)^{3/4}\right] \tag{22}$$

Therefore,

$$r = \left(\frac{13}{14}\right)^{1/3}\left(\frac{\alpha}{v}\right)^{1/3}\left[1-\left(\frac{x_o}{x}\right)^{3/4}\right]^{1/3}$$

$$r = \frac{0.976}{\Pr^{1/3}}\left[1-\left(\frac{x_o}{x}\right)^{3/4}\right]^{1/3} \tag{23}$$

If heating of the plate starts from the leading edge of the plate, then x_o=0. Equation (23) becomes

$$r = \frac{0.976}{\Pr^{1/3}} \tag{24}$$

The local heat transfer coefficient can be determined as

$$\frac{Q}{A} = h_x\left(T_s - T_f\right) = -k\left(\frac{dT}{dy}\right)_{y=0}$$

$$h_x = \frac{-k\left(dT/dy\right)_{y=0}}{\left(T_s - T_f\right)} \tag{25}$$

Substituting the value of $\left(dT/dy\right)_{y=0}$ from equation (17) in to equation (25)

$$h_x = -\frac{3k}{2\delta_t}\frac{\left(T_f - T_s\right)}{\left(T_s - T_f\right)} = \frac{3k}{2\delta_t} = \frac{3k}{2r\delta} \tag{26}$$

We know that

$$\delta = \frac{4.64\,x}{\sqrt{\mathrm{Re}_x}} \tag{27}$$

Substituting the values of δ from equation (27) and r from equation (23) in equation (26)

$$h_x = \frac{3k}{2}\frac{\sqrt{\mathrm{Re}_x}}{4.64x} \times \frac{\mathrm{Pr}^{1/3}}{0.976\left[1-\left(x_o/x\right)^{3/4}\right]^{1/3}}$$

$$h_x = 0.332\frac{k}{x}\left(\mathrm{Re}_x\right)^{1/2}\left(\mathrm{Pr}\right)^{1/3} \times \frac{1}{\left[1-\left(x_o/x\right)^{3/4}\right]^{1/3}} \tag{28}$$

Nusselt Number can be expressed as

$$Nu_x = \frac{xh_x}{k} = 0.332\left(\mathrm{Re}_x\right)^{1/2}\left(\mathrm{Pr}\right)^{1/3} \times \frac{1}{\left[1-\left(x_o/x\right)^{3/4}\right]^{1/3}}$$

If the entire length of the plate is heated, x_o=0

$$Nu_x = 0.332\left(\mathrm{Re}_x\right)^{1/2}\left(\mathrm{Pr}\right)^{1/3} \tag{29}$$

$$h_x = 0.332\frac{k}{x}\left(\mathrm{Re}_x\right)^{1/2}\left(\mathrm{Pr}\right)^{1/3} \tag{30}$$