LESSON 9

Special cases

General governing equation for heat transfer from a finned surface is expressed as

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (1)$$

Soultion of equation (1) is expressed as

$$\theta = c_1 \cosh mx + c_2 \sinh mx \qquad (2)$$

Using the above two equations, following special cases are considered for heat transfer through a fin of uniform cross section:

1. Fin is losing heat at the tip only

When a fin of finite length loses heat only at its tip as shown in Figure 1, the relevant boundary conditions are

i) **At the base of fin, its temperature is equal to the wall temperature. Therefore,**

At $x=0$; $T = T_0$

$T - T_a = T_0 - T_a$

or $\theta = \theta_0 \qquad (3)$

θ_o represents temperature difference between the fin base and the fluid surrounding the fin.

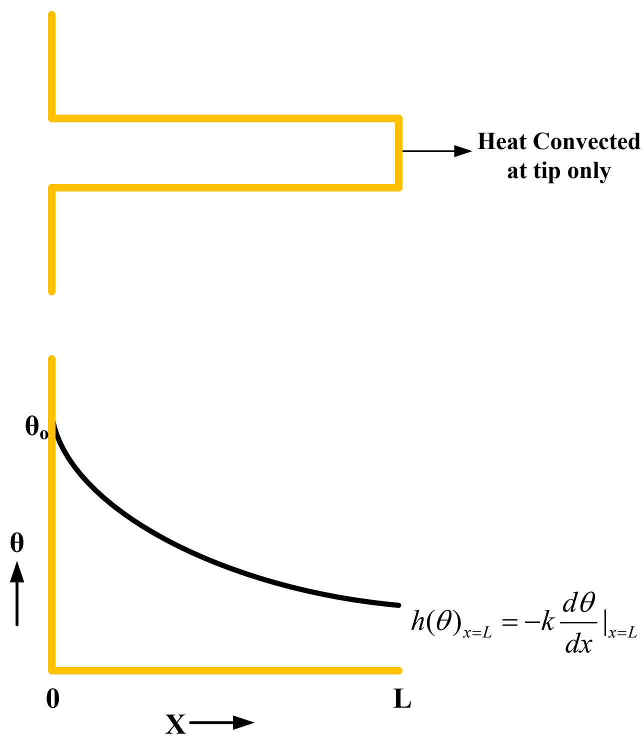

Figure 1

Applying first boundary condition to equation (2), we get

$\theta_o = c_1 \cosh m\times 0 + c_2 \sinh m\times 0$

$\Rightarrow \quad C_1 = \theta_o$ (4)

ii) **As fin is losing heat at tip only, it means heat conducted through fin is lost to surrounding fluid by convection at tip. Therefore,**

At $x = L$;

Heat conducted through fin = Heat convected to surrounding fluid by convection

$$-kA_c \frac{dT}{dx}|_{x=L} = hA_c(T_{x=L} - T_a)$$

$$-k\frac{d\theta}{dx}|_{x=L} = h(\theta)_{x=L} \quad (5)$$

Substituting the value of C_1 from equation (4) in equation (2), we get

$$\theta = \theta_o \cosh mx + c_2 \sinh mx \quad (6)$$

Differentiating equation (6) with respect to x, we get

$$\frac{d\theta}{dx} = \theta_o m \sinh mx + mC_2 \cosh mx \quad (7)$$

Substituting value of $\frac{d\theta}{dx}$ from equation (7) in equation (5), we get

$$-k(\theta_o m \sinh mx + mC_2 \cosh mx)_{x=L} = h(\theta)_{x=L}$$

Substituting value of 'θ' from equation (6) in equation (5), we get

$$-k(\theta_0 m \sinh mL + c_2 m \cosh mL) = h(\theta_0 \cosh mL + c_2 \sinh mL)$$

Solving above equation for value of C_2, we can write

$$c_2 = \frac{-\theta_0(mk \sinh mL + h\cosh mL)}{(h\sinh mL + mk\cosh mL)} \quad (8)$$

Substituting the values of C_1 and C_2 in equation (6) we get

$$\theta = \theta_0 \cosh mx - \theta_0 \frac{(mk\sinh mL + h\cosh mL)}{(h\sinh mL + mk\cosh mL)}\sinh mx \quad (9)$$

Equation (9) represents temperature distribution.

Rate of heat transfer from fin is expressed as

$$Q = -kA_c \frac{dT}{dx}|_{x=0} = -kA_c \frac{d\theta}{dx}|_{x=0} \qquad \text{as} \quad \frac{dT}{dx} = \frac{d\theta}{dx}$$

Therefore, $$Q = -kA_c \frac{d\theta}{dx}|_{x=0} \tag{10}$$

Differentiating equation (9) with respect to x, we get

$$\left(\frac{d\theta}{dx}\right) = \theta_o m \sinh mx - \theta_o m \left(\frac{h\cosh mL + mk\sinh mL}{mk\cosh mL + h\sinh mL}\right)\cosh mx$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = 0 - \theta_o m \left(\frac{h\cosh mL + mk\sinh mL}{mk\cosh mL + h\sinh mL}\right) \tag{11}$$

Substituting the value of $\left(\frac{d\theta}{dx}\right)_{x=0}$ from above equation in equation (10), we get

$$\Rightarrow Q = kA_c\theta_0 m\left(\frac{mk\sinh mL + h\cosh mL}{h\sinh mL + mk\cosh mL}\right) \tag{12}$$

Equation (12) can be written as

$$Q = kA_c\theta_0 m\left(\frac{\sinh mL + \frac{h}{mk}\cosh mL}{\cosh mL + \frac{h}{mk}\sinh mL}\right) \tag{13}$$

Substituting the value of $m = \sqrt{\frac{hP}{kA_c}}$ and $\theta_0 = T_0 - T_a$ in above equation, we get

$$Q = \sqrt{PhkAc}\left(T_o - T_a\right)\left(\frac{\sinh mL + \frac{h}{mk}\cosh mL}{\cosh mL + \frac{h}{mk}\sinh mL}\right) \tag{14}$$

Multiplying and dividing the left hand side of above equation by coshmL, we get

$$Q = \sqrt{PhkAc}\left(T_o - T_a\right)\left(\frac{\tanh mL + \frac{h}{mk}}{1 + \frac{h}{mk}\tanh mL}\right) \tag{15}$$

The equation gives represents rate of heat transfer from a fin which is losing heat at the tip only.

2. Fin is sufficiently long or infinite

For a fin of infinite length as shown in Figure 2, the relevant boundary conditions are:

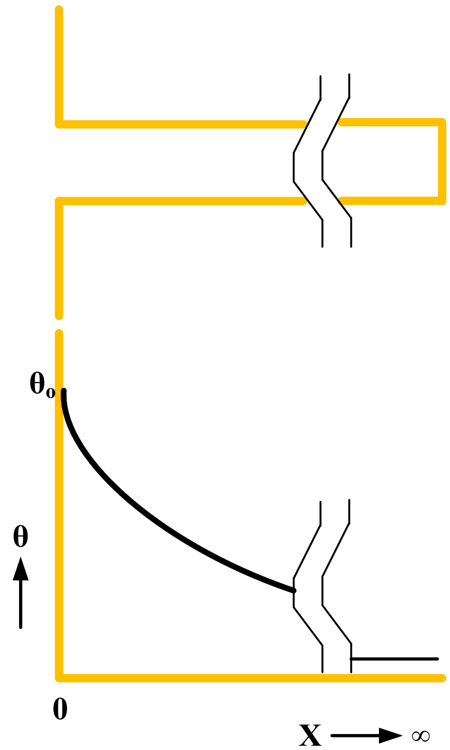

Figure 2

i) At the base of fin, its temperature is equal to the wall temperature. Therefore,

At $x=0$; $T = T_0$

$T-T_a = T_0 - T_a$

or $\theta = \theta_0$ (16)

θ_o represents temperature difference between the fin base and the fluid surrounding the fin.

ii) For a sufficient or infinitely long fin, temperature at the tip of fin is equal to that of surroundings.

At $x=L$, $\theta=0$ (17)

Applying the first boundary condition to equation (2), we get

$\theta_o = C_1 \cosh m\times 0 + C_2 \sinh m\times 0$, we get

$C_1 = \theta_0$ (18)

Applying second boundary condition to equation (2), we get

$0 = \theta_o \cosh mL + C_2 \sinh mL$

$$C_2 = -\theta_o \coth mL \tag{19}$$

Substituting the values of C_1 and C_2 in equation (2)

$$\theta = \theta_o \left(\cosh mx - \coth mL \sinh mx\right) \tag{20}$$

Rate of heat transfer from fin at base or root is expressed as;

$$Q = -kA_c \frac{dT}{dx}|_{x=0} = -kA_c \frac{d\theta}{dx}|_{x=0} \tag{21}$$

Differentiating equation (20) with respect to x, we get

$$\left(\frac{d\theta}{dx}\right) = \theta_0 \left(m \sinh mx - \coth mL \times m \cosh mx\right)$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = \theta_0 \left(m \sinh m \times 0 - \coth mL \times m \cosh m \times 0\right)$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = -\theta_0 m \coth mL \tag{22}$$

Substituting the value of $\left(\frac{d\theta}{dx}\right)_{x=0}$ in equation (21), we get

$$Q = kA_c \theta_o m \coth mL$$

As $L \to \infty, \coth mL \to 1$

$$\therefore Q = kA_c m\theta_0 \tag{23}$$

Equation (23) represents rate of heat transfer from a fin of infinite length.

3. Fin is insulated at the tip

For a fin of finite length having its end insulated, no heat transfer takes place from the tip of the fin as shown in Figure 3. The relevant boundary conditions are:

i) At $x = 0; \theta = \theta_0$

ii) At $x = L; \frac{dT}{dx} = 0 \Rightarrow \frac{d\theta}{dx} = 0$

Applying the first boundary conditions to equation (2), we get

$$C_1 = \theta_0 \tag{24}$$

Applying second boundary condition to equation (2), we get

$$\left(\frac{d\theta}{dx}\right)_{x=L} = m\left(\theta_0 \sinh mL + C_2 \cosh mL\right) = 0$$

$$C_2 = -\theta_0 \tanh mL \quad (25)$$

Substituting the values of C_1 and C_2 in equation (2), we get

$$\therefore \theta = \theta_0 \cosh mx - \theta_0 \tanh mL \sinh mx \quad (26)$$

Equation (26) represents temperature distribution in a fin having its end insulated.

Rate of heat transfer from fin at base or root is expressed as;

$$Q = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c \frac{d\theta}{dx}\Big|_{x=0}$$

$$Q = kA_c\theta_0 m \tanh mL \quad (27)$$

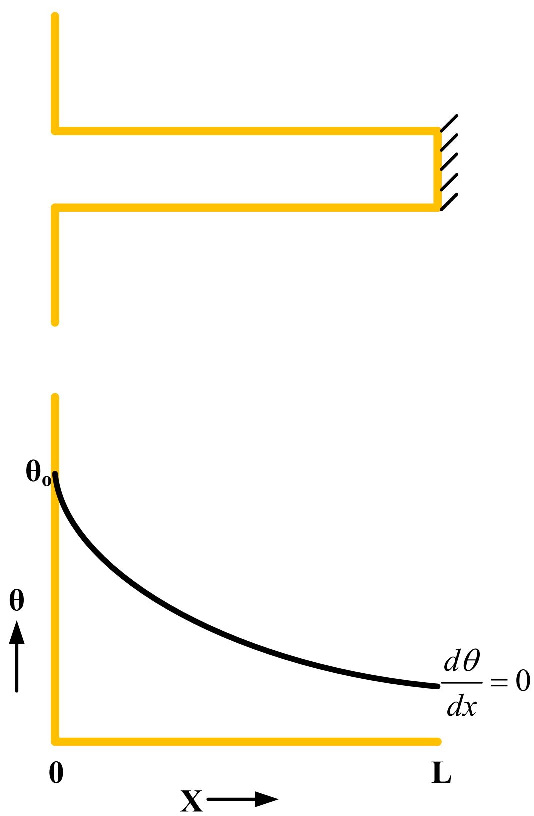

Figure 3

REVIEW QUESTIONS:

1. For a finned surface, it is considered appropriate that area of cross-section be
   a) maintained constant along the length
   b) increased along the length
   c) reduced along the length
   d) it is better to vary the convection coefficient than the area

2. An increase in convective coefficient over a fin
   a) increases effectiveness
   b) decreases effectiveness
   c) does not influence effectiveness
   d) influences only the fin efficiency

3. An increase in fin effectiveness is caused by high value of
   a) convective coefficient
   b) thermal conductivity
   c) sectional area
   d) circumference

4. Fin efficiency is defined as the ratio of the heat transferred across the fin surface to the theoretical heat transfer across an equal area held at
   a) temperature of fin end
   b) constant temperature equal to that of base
   c) average temperature of fin
   d) none of the above

5. Consider a square section fin split longitudinally and used as two fins. This will result in
   a) increase in heat transfer
   b) decrease in heat transfer
   c) increase or decrease in heat transfer depending on material of fin

d) heat flow remain constant

6. Mark the false statement regarding effectiveness of fin:

a) fin effectiveness represents the ratio of heat dissipation with a fin to the heat transfer that would exist without a fin.

b) fin effectiveness represents the ratio of heat transfer rate from the fin to the heat that would the dissipated if the entire fin surface area were maintained at the base temperature.

c) fin effectiveness is improved if the fin is made from a material of low thermal conductivity.

d) a high value of film coefficient adversely affects the fin effectiveness.

e) fin effectiveness is improved by having thin but closely spaced fins.

7. The heat dissipation from an infinitely long fine is given by

a) $\sqrt{\mathbf{phkA_c}}\,(\mathbf{t_0 - t_a})$

b) $hpl\,(t_0 - t_a)$

c) $\sqrt{phkA_c}\,(t_0 - t_a)$ tanh ml

d) $\sqrt{phkA_c}\,(t_0 - t_a)\,\frac{\tanh ml + (h/k)m}{1 + (h/km)\tanh ml}$